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Common Cost Allocations and Game Theory : A New Approach

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Abstract

Some game theoretical solutions (*Shapley value, nucleolus,* and so on) are proposed as allocation schemes in common costs allocation. Game theoretical solutions have desired properties compared with the conventional allocation scheme based on some allocation bases. Many authors consider that cooperative game solutions, which might be rather complex and difficult to formulate than the conventional method, become new allocation schemes. But it seems that the conventional allocation scheme is widespread in practice. A new approach to common costs allocation is proposed to examine this reason.

We examine three simple models under the proposed approach. They are formulated with the two-person game and describe the allocation practice appropriately. The results obtained from these models indicate that the conventional allocation method does not have serious problems in two models but have difficulties in one model. Whether the conventional allocation method has difficulties or not in the common costs allocation setting depends on the information each department has. Therefore it is important to identify the situation where the common costs allocation is necessary when the conventional allocation method is used as an allocation scheme.

Key Words

Cost Allocation, Common Costs, Game Theory, Coalition, Two-Person Game

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1. *Introduction*

Many authors tried to apply game theoretical solutions to common cost allocations in the 1970's and 1980's. They proposed *Shapley value, nucleolus* and the others as new allocation schemes in their studies. These solutions seem to have desirable properties compared with the conventional allocation method based on allocation bases. Because they give service departments' users more satisfactory and fair solutions than the conventional allocation method. In spite of the game theoretical solutions' desirable properties, these new schemes are not widespread in practice. In this article, we would like to examine the reason the game theoretical solutions are not popular as the common cost allocation schemes in practice.

We propose a new approach to the common costs allocation to attain this purpose. This approach is based on the two-person game not on the characteristic function form game. We examine the implications of the game theory in common cost allocation with this proposed approach.

In the next section, we define the common costs allocation and examine the implication n-person characteristic function form game in the common costs allocation. In the third section, we clarify the game theoretical approach in common cost allocation used by past studies and propose a new approach. We present one proposition that is verified by the examination in the later section.

We describe the three departments model in the fourth section. The formulation of the model is based on the approach proposed in Section 3. We examine the three cases in Section 5. First case and second case are different in the assumption of the coalition formation. Second case and third case are different in the assumption of the information departments have. We obtain the result that the conventional allocation method is justified in the first and second case. In the last section, we summarize the results in this article and present the related topics to be resolved in the future.

2. *Common Cost Allocations and Game Theory*

2.1 Common Cost

There were many treaties on common cost allocation in which game theoretical solutions are proposed as allocation schemes. The term "common cost" has a broad definition and its use is likely to be confusing. It is necessary to define the term "common cost" in this article to make later discussions clear. For this definition we refer to Biddle and Steinberg[1984], in which they surveyed common cost allocation comprehensively. [1]

[1] Biddle and Steinberg [1984], p.5.

> Common cost applies to a setting in which production costs are defined on a single intermediate product or service that is used by two or more users.

The point in this definition is that two or more users use the single intermediate products or services jointly. The users of services (divisions or departments in a firm) expect to receive some benefits from obtaining the necessary services jointly. In other words, the users consider the joint acquisition of the services for their cost saving. Therefore it is implied that there are some joint benefits in the common costs allocation. An allocation method is necessary to allocate these benefits among the users of services with some satisfactory manners.

Sometimes, the conventional allocation method based on some allocation bases cannot allocate the cost saving from joint acquisitions properly. Moriarity[1975] points out the difficulties inherent in the conventional method by giving some examples. A characteristic function form game is proposed for settling such difficulties.

2.2 Characterisitc Function Form Game in Common Cost Allocations

Many studies, in which game theoretical solutions are applied into the common costs allocation, formulate the common costs allocation setting as a characteristic function form game. [2] In their formulation, managers of divisions or departments are regarded as players of a game and players can form any coalitions to gain joint benefits.[3] As we examine the common cost allocation setting to which a characteristic function form game is applied in this article, it is important for our discussion to consider the implication of this formulation.

Strictly speaking, there are two types of the characteristic function, namely the transferable utility characteristic function and the characteristic function for a non-transferable utility game.[4] When a common cost allocation setting is formulated as a characteristic function form game, it is proper to assume the existence of the transferable utility. If the existence of the transferable utility is not admitted, it is impossible to share some cost saving gained by a coalition among players. So it is useful to examine the implications of this assumption and to confirm that this assumption is rational in common cost allocations.

Luce and Raiffa[1957] explain the cases where the assumption of transferable utility is rational. [5]

[2] See Owen [1982] (Chapter 8) for the characteristic function form game.
[3] A coalition is a subset of the set of all players. Refer to Owen [1982] (p.145) for the strict definition of a coalition.
[4] Refer to Friedman [1990] (p.244,277) for the definition of these characteristic functions respectively.
[5] Luce and Raiffa[1957], p.181. These conditions are sufficient not necessary for the existence of the transferable utility.

- Monetary side payments are allowable.
- Each player's utility for money is approximately linear in the range of potential payoff of the game.

It is clear what the first condition means in common cost allocations. The second condition says that there are no extreme differences in the allocated amount to each department. As it is natural that we discuss common costs within the relevant range, the second condition also holds in the common cost allocation settings. So we proceed with the discussions based on the assumption of the transferable utility in this paper.

Friedman[1990] comments on the transferable utility.

> If the players in the game are firms in a market, it may seem reasonable to assume transferable utility on the ground that income in money measures utility for each firm, and does so in same way for all firms.(Friedman[1990], p.242.)

It seems that his comment applies to the departments or divisions in a firm. We consider the amount of cost as the measure proxy to the transferable utility in the later discussions.

Next, we examine the implication of the characteristic function in the context of common cost allocations. Owen[1982] defines the characteristic function as follows:[6]

> By the characteristic function of n-person game we mean a real-valued function v defined on the subset of N, which assigns to each $S \subset N$ the maxmini values (to S) of the two-person game played between S and $N - S$, assuming that these two coalitions form.

This definition tells us that the following three points should be made clear when a common cost allocation setting is formulated as a characteristic function form game.

- The set of players.
- Coalition.
- How to estimate a relevant characteristic function.

We have no difficulties in regarding the departments as the players of the game.[7] It means that each department can make decisions for obtaining its own services.

A coalition is the subset of all players. It is necessary for characteristic function form formulation to guarantee that departments can form a coalition unrestrictedly in

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[6] Owen[1982], p.145. The value of v is utility or benefit in this definition. If a characteristic function is a cost based one, "minimax" is substituted for "maxmin."

[7] Strictly speaking, the departmental managers are the players of the game. But we use the term "departments" instead of "departmental managers", for some abbreviation in this article.

order to get necessary services. It means that departments in the same coalition can make decisions jointly for obtaining services.

When a value is estimated for coalition S, two-person game is played between coalition S and $N-S$ and the minimax value is assigned to coalition S. This idea is essential to this article although no characteristic function is estimated in later section. Because later analyses are based on this idea.

As we accept the assumption of the transferable utility and interpret the characteristic function in the context of the common cost allocation, we can specify the case that we will examine in this article.

- There are some external vendors in the market that provide the departments with compatible services provided internally.

- The departmental managers prefer lower allocated costs, because they think that the allocated cost to their departments relates to their performance evaluation.

- Departmental managers can decide whether they get the internal service or the external service and can negotiate about the coalition formation with the other departments.

The first condition assures that feasibility of the external acquisition of the service. Without this, department do not have the incentive to form a coalition. The second and third conditions mean that a department always looks for the alternatives to get the necessary service inexpensively.

3. Game Theoretical Approach to Common Cost Allocation

3.1 An Approach in the Past Studies

If an allocation scheme gives the users of service departments acceptable amount, it is a desirable allocation scheme. In other words, a desirable allocation scheme yields satisfactory allocations to all users.[8] When such a desirable allocation scheme is used, the departments do not have the incentives to obtain necessary services from the external vendors even if they could get the services with lower cost.

When the conventional allocation method is used as an allocation scheme, it is likely that the departments are not satisfactory for the amount allocated to them. Consequently, the departments may make suboptimal decisions in terms of a firm as a whole. Some game theoretical solutions are proposed to decrease the dissatisfactions of the departments with their allocated cost. Therefore game theoretical solutions are regarded as a means yielding accounting information that dissuade the

[8] Such an allocation is called "mutually satisfactory allocation" in Thomas[1974].

departments from making suboptimal decisions.

There are many studies which try to apply game theoretical solutions to the common costs allocation problem.[9] The approach in these studies are summarized as three steps described below.

1. Specifying the situation where common cost allocations are necessary.
2. Making clear the desired properties of the allocation schemes in common cost allocations.
3. Searching for some game theoretical solutions which satisfy the properties specified above.

The approach is expressed as the following Figure 1.

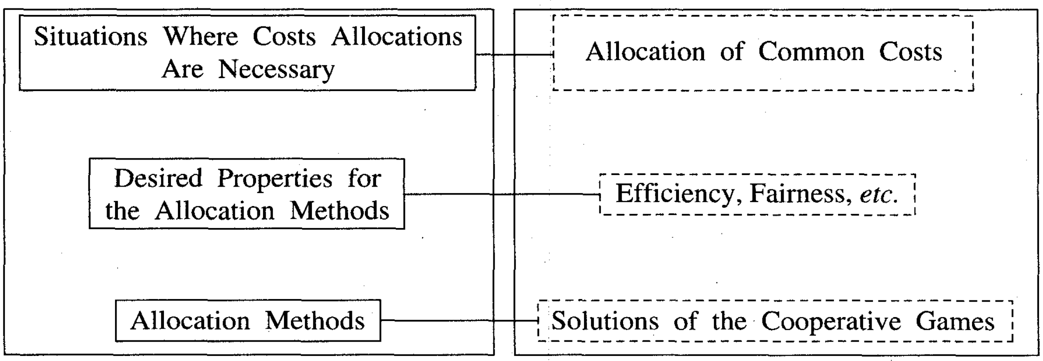

Figure 1 : The Past Study Approach

A common cost allocation setting is formulated with a characteristic function in the first step, so it is necessary to estimate a characteristic function properly in this step. Every department knows the estimates of the characteristic function and accepts these values as a base for cost allocations. It is an interesting topic to estimate some characteristic function in the common cost allocation setting. As this topic is not the main purpose of this article, we do not address this topic in detail here.[10]

In the second step, it is noted that the concept of fairness (equity) is included as desirable properties in the allocation scheme. Allocation bases are related to common costs implicitly in the conventional allocation scheme. So it allocates common costs in proportion to allocation bases. In some cases, such allocation does not give satisfactory results to departments as Moriarity[1975] pointed out.

The conventional allocation method does not incorporate the concept of fairness in itself. Selecting proper allocation bases is the most important in this method. Whether this method is satisfactory to department or not depends on the choice of allocation base. In contrast to this, it is relatively easy to give a game theoretical solution some meaningful interpretations in terms of the desired properties. Various

[9] See Biddle and Steinberg[1984] for the comprehensive list of game theory applications to common costs allocations.
[10] There are many kinds of a characteristic function, *i.e.*, cost based one,benefit(cost saving) based one, profit based one, and so on.

game theoretical solutions are selected as the allocation schemes in the third step.

When we use the approach described above, the main purpose of the study is to explain the relationships between the concept of the game theoretical solutions and the desired properties in common cost allocations rationally. Namely, interpreting the game theoretical solution in the context of common cost allocation is our main purpose. It enables the departments to understand the implications of the game theoretical solutions and to accept them as the desirable allocation schemes. This is explained by using *Shapley value* and *nucleolus* as examples , which are familiar game theoretical solutions in the past studies.

Shapley value is derived from three axioms uniquely.[11] When *Shapley value* is applied to common cost allocations, it is necessary to interpret these axioms in terms of the properties desired in common cost allocations. *Shapley value* is based on the idea in which incremental benefits (or costs) are allocated among players evenly. This idea is deemed to be fair or equitable. Shubik[1962], Loehman and Whinston[1974], and Jensen[1974] deal with the interpretations of *Shapley value's* axioms. Roth and Verrecchia[1979] also interpret *Shapley value* axioms in the context of a bargaining setting.

Nucleolus is based on the principle "minimizing the maximum surplus." [12] As this principle is similar to the Rawls' justice concept, it is relatively easy to interpret this principle on Rowls' justice concept and for departmental managers to accept it.[13]

Nucleolus give a unique core solution whenever core exists.[14] It is another desirable property of *nucleolus*. So in this sense, *nucleolus* yield a solution that does not induce suboptimal decisions of the departments. Hamlen *et al.*[1977] mentioned that *nucleolus* is most desirable compared to another allocation schemes because it yields unique core allocation. As there are some kind of measures denoting *nucleolus* surplus, it is possible to focus on these measures and this is another direction of the study.[15]

3.2 A New Approach to Common Cost Allocations

The conventional method based on some allocation bases, which is widespread in practice, has some difficulties in that this method induces the departments to make suboptimal decision making. Game theoretical solutions are proposed to overcome such difficulties. Game theoretical solutions may have the possibility of solving such

[11] It is necessary to define a carrier in addition to three axioms in order to derive *Shapley value*. Refer to Owen[1982](pp.193-194) for Shapley axioms and a carrier. See also Shapley[1953] for the original derivation of *Shapley value*.

[12] The surplus is defined by the excess of coalition with respect to the payoff vector. Refer Owen[1982] for the excess and the surplus. See Schmeidler[1969] for nucleolus.

[13] See Rawls[1958].

[14] See Owen[1982](pp.244-256) for the properties of *nucleolus*. See Owen[1982](pp.150-164) for the core and its properties.

[15] See Gately[1974], Littlechild and Vaidya[1976], Charnes *et al.*[1978], and Young *et al.*[1980] for a variety of *nucleolus*. Aoki[1988] examines these *nucleolus* in the common costs allocation. Recently Sobolev[1995] examines the axiomatization of *nucleolus*.

difficulties and may give us one breakthrough to make new cost allocation information in accounting. Therefore, studying game theoretical solution concept and applying it to the cost allocation problems might be important.

Game theoretical solutions as the allocation scheme have many desirable properties as the allocation compared to the conventional allocation method. Because they give fair or equitable solution to the users of service departments. It is natural that game theoretical solutions are applied to common cost allocations in practice because of their desirable properties, but they are not used in many cases. Why?

Certainly there are some technical difficulties for applying game theoretical solutions to common cost allocations. Here, we consider two difficulties.

- It is cumbersome and difficult to estimate a characteristic function in the common cost allocation setting rationally.

- Complex computations are necessary to get some game theoretical solutions, so much time and cost are consumed to get them.

When a characteristic function is estimated, it is necessary to specify the benefits arising from the joint acquisition of the services and to quantify these benefits for all possible coalitions.[16] If the vendors supplying the necessary services exist in the external market, estimating a characteristic function may be relatively easy. Because we can compare the internal cost data with the external cost data.

If departments cannot get the necessary services from the external market, the estimation of a characteristic function may be troublesome. But it might be possible to estimate a characteristic function using the cost function derived from internal cost data. Though it is a curious topic to study, it is not the main purpose of this article.

The second difficulty described above is not serious compared with the first difficulty. Because the performance of computers are now increasing drastically, it may be necessary to improve computation algorithm for deriving game theoretical solutions.[17]

The two difficulties described above are technical problems. Thus it may be to overcome these difficulties by something new inventions. It is appropriate that these difficulties are not intrinsic to game theoretical solutions in the common costs allocation. We must think the essential reason that game theoretical solutions are not used in practice.

There may be many ways to explain this reason. One simple way is to change our recognition for the conventional allocation method. Namely, we examine the common cost allocation setting based on the understanding that the conventional allocation method does not have serious problems. We go on our analysis in this article under the following proposition.

[16] If there are n departments, 2^n-1 estimates of a characteristic function are required. The number of estimates increases rapidly as the number of departments increases.

[17] See Littlechild and Owen[1973], Littlechild[1974], Suzuki and Nakayama[1976], and Legros[1986].

Proposition: Cost accountants do not consider the conventional allocation method as one giving rise to serious problems in common cost allocations. Even if they recognize difficulties, they avoid these by another approach, which is not a search for some new allocation schemes.

This proposition means that cost accountants cannot avoid using the conventional allocation method in the present accounting framework. This proposition also does not eliminate the fact that some problems remain in the conventional allocation method. If we accept this proposition, we can analyze the common cost allocation more practically than the approach that is searching for new allocation schemes.

Based on the above proposition, we approach to the common cost allocation with the conventional method. So we do not look for a new allocation scheme in this article. Under this approach, game theory is a tool for analyzing the behavior of the departmental managers(players) in the common cost allocation settings not a tool for deriving a new allocation method. Namely, game theory is used to analyze the decision processes of managers given some allocation amount by the conventional method. As players of the game decide their strategy based on their information, information, especially cost information, plays an important role under the approach presented here.

The approach in this article is summarized as Figure 2.

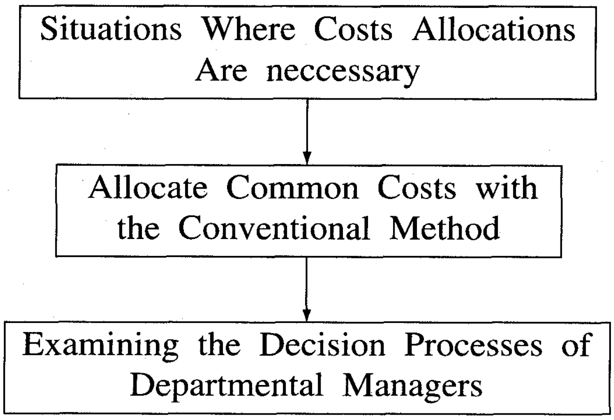

Figure 2 : The Approach in this Paper

Departments must have cost information for their decisions when they obtain the service. This cost information is specified in the first step. Common costs are allocated with the conventional allocation method in the second step. It means that the conventional allocation method is incorporated into the allocation game as a rule. This point is very important and essential under the approach in this paper. The behavior of departments is examined in terms of information in the third step.

4. *Model Description*

4.1 The Purpose of the Model

We formulate the situation that is specified in Section 2.2 and examine this with the approach proposed in Section 3.2. We use a case where there are one service department and three operating departments using the service in a firm. Suppose that there are external vendors that provide the operating departments with the service equivalent to the internal service.

It is supposed that the departments(game players) know the rule for calculating their allocated costs. When they get the necessary service from the service department, service department costs are allocated to users in proportion to some allocation bases. The price of the service in the external market is given by the function defined later. When departments get the external service jointly, the cost of service is also prorated according to some allocation bases. Namely, The conventional allocation method is considered as the rule of a game in this analysis.

We will not analyze the case where the conventional allocation method yields a stable solution certainly. Therefore we will not deal with the case where the minimum cost for players or coalitions is concave. Because the conventional allocation method gives a core solution in this case. Thus players do not have any positive reasons to object this allocation. We examine the situation where the minimum cost for players or coalition is not smooth but skewed in this article. In the later, we propose the case where the conventional allocation method may induce the departments to make some suboptimal decisions in terms of the firm as a whole. Therefore, the situation studied in this article may not be general.

We scrutinize the behaviors of the departmental managers with the proposed model. The purposes of the analysis are to examine whether the conventional allocation method gives departments the incentive to get the external service or not. If we can specify the conditions in which the conventional allocation method does not have any difficulties, it means that Proposition in Section 3.2 is justified under these conditions.

4.2 Three Departments Model

We treat the case where there are one service department and three operating departments. Operating departments are players of the game, *i.e.*, $N = \{ 1, 2, 3 \}$. S $(\subset N)$ is a coalition of the departments.

Each departmental manager has two alternatives for the acquisition of the service. One is internal acquisition and the other is external acquisition. These alternatives are called strategies and have the following notations.

- I_i : the strategy in which department i gets the service internally. ($i \in N$)
- O_i : the strategy in which department i gets the service externally. ($i \in$ N)

The cost functions and some abbreviations are defined.

- q_i (> 0) : the amount of the service that department i demands. ($i \in N$)
- $c\ (q)$: the cost function for the internal service. q is the amount of the service.
- $f\ (q)$: the cost function for the external vendor. q is the amount of the service.
- $q\ (S) = \sum_{i \in S} q_i$: the amount of the service for coalition S. We will use this notation as abbreviation for summation occasionally.
- $\bar{q} = q\ (N)$: the total amount of the service.

It is assumed that $c\ (q)$ is a monotonic increasing and concave function. It means that the scale of economy works when departments use the internal service. It is assumed that $f\ (q)$ is a monotonic increasing and convex function. For example, consider the situation where the more departments get the external service, the more costs, such as the expense of the acceptance, are incurred.[18] All departments have information about these functions in common.

According to the assumption in the previous section, common costs are always allocated to each department proportionally to the allocation base. We regard q as the allocation base in this analysis. During the planning the next year's budget, each department sends the information about q_i to the section (*i.e.*, accounting division) in which the budgeted burden rate is calculated. Assuming all departments use the internal service, this section calculates the budgeted burden rate $r = c\ (\bar{q})\ /\ \bar{q}$. The resulting information about r is send to all departments.[19]

The relationship between the cost information about the external service $f\ (q)$ and the cost information about the internal service r plays an important role in the models of this article. Because it is supposed that departments decide their strategy by considering. these information. This relationship is presented with the notations defined above.[20]

$$f\,(q\,(S)) < rq\,(S) \qquad (\forall S \subset N \ \ for\ \ |S| \leq 2\,) \tag{1}$$

$$f\,(\bar{q}\,) > c\,(\bar{q}\,) \tag{2}$$

The first condition says that a department or a coalition consisting of two departments can get the external service at a lower cost than the internal service. It means

[18] This assumption seems to be strange because the average costs of the external service decreases if there are some discount factors for the service. But it is supposed that the amount of some discounts is lower than the miscellaneous costs incurred by the external acquisition.

[19] As r is the predetermined rate, some variances will occur undoubtedly if the common cost is allocated with this rate. Including these variances into the model makes the model complex and is not the intention of this article. So we consider r as a constant.

[20] $|S|$ is the number of the departments in a coalition S.

that all departments have the incentive to get the service from the external vendor. This condition also denotes that the conventional allocation method yields an unstable allocation.

The second condition says that an economically inefficient situation occurs if all departments obtain the service externally. It is not necessary to specify the correct value of $\bar{q}$ in this assumption. The point of this assumption is the fact that external costs of the service is greater than the internal costs of the service when a grand coalition is formed. It is supposed that all departments know the relationship described in (1) and (2).[21]

The above relationship is depicted in Figure 3.

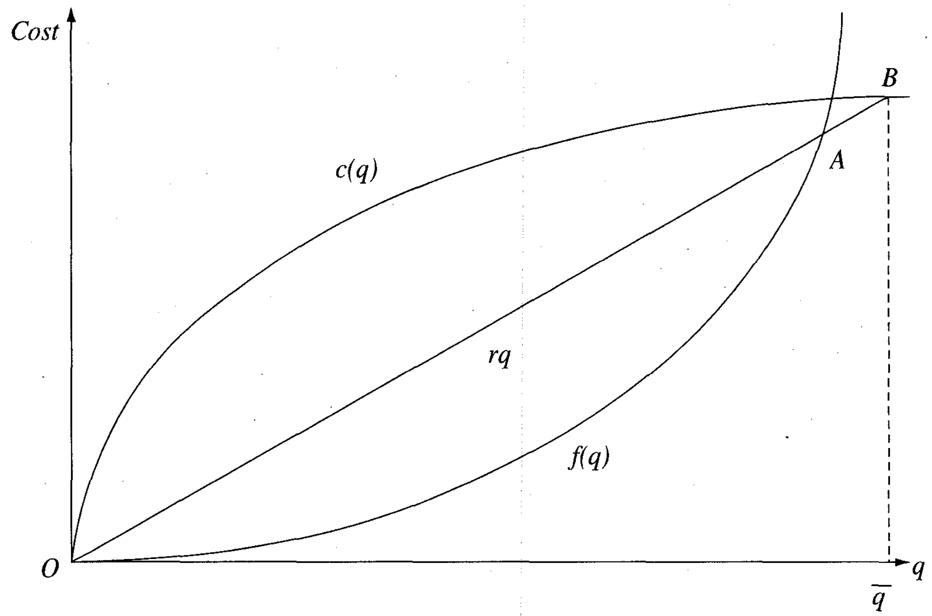

Figure 3.: Cost Function

It is noted that a line OB represents the allocated costs not the amount of the incurred costs while $c(q)$ and $f(q)$ represent the amount of the costs to be incurred. As every department tries to get the service at the lowest cost, it regards the curve $\widehat{OAB}$ as the service cost information.

The minimum cost of the service as a firm is supposed to be $c(\bar{q})$. Because the meaning of the budgeted burden rate r is lost if there is some coalition structure $\{S, N{-}S\}$ satisfying $f(q(S)) + c(q(N-S)) < c(\bar{q})$. Thus the following inequality is supposed.

$$f(q(S)) + c(q(N{-}S)) \geq c(\bar{q}) \qquad (\forall\, S \subset N) \tag{3}$$

As each department obtains the information about the internal cost $c(\bar{q})$ through r indirectly, it is natural that it does not know the condition (3). Accordingly, depart-

[21] If $f(q)$ is a concave function, it is difficult to assume these conditions.

ments do not regard $c(\bar{q})$ as the minimum cost as a firm since the following inequality may hold from (1) and (2).

$$f(q(S)) + rq(N-S) < c(\bar{q}) \quad (\exists S \subset N) \tag{4}$$

This inequality suggests that coalition S gets the external service and coalition $N-S$ gets the internal service. But the minimum cost is not achieved in this case because of (3). Some idle costs exist in this case. This is a difficulty of the conventional allocation method.

The purpose of the analysis in this article is to denote that the situation described above is not always true. We focus on the coalition structure other than $\{N\}$ and information each coalition has.[22]

5 Model and Examination

5.1 Model 1: Each Department Makes Its Decision Independently

Model 1 describes the case where departments do not negotiate for the acquisition of the service with other managers. It is assumed that each department makes a decision for its acquisition of the service independently, *i.e.*, getting the internal service or getting the external service. The decision process of department 1 is examined in the later analysis but the results of the analysis apply to other departments.

It is supposed that department 1 (D1) does not have any information about other departments in Model 1.[23] Hence D1 does not know other department's service amount, namely q_2 and q_3. D1 cannot predict the response of other departments to its strategy because of the lack of information. D1 only knows the possible combination of the strategies that other department may choose, namely I_2I_3, I_2O_3, O_2I_3, O_2O_3.

For convenience, D2 and D3 are regarded as one player who has the above four strategies and select these strategies at random.[24]

This player is called "D23". Note that D2 and D3 make their decision independently and do not form a coalition in Model 1.

If D1 selects the strategy I_1, the allocated cost to D1 is rq_1. This amount does not depend on D23's strategy. But if D1 selects the strategy O_1, what D1 can estimate for certain is the cost corresponding to the strategy I_2I_3. Its amount is $f(q_1)$. It is necessary for D1 to know q_2 and q_3 in order to estimate the costs corresponding to the strategies other than I_2I_3. As D1 cannot specify the correct values corresponding to I_2O_3, O_2I_3, and O_2O_3, we denote these elements as a_{22}, a_{23}, and a_{24} instead of using some formulas in the next payoff matrix.

[22] A coalition structure is a partition of the set of players. Refer to Owen [1982], p.236.
[23] We abbreviate department 1 as D1 in the later. So D2, D3 and so on have the same abbreviation.
[24] Department 23 is a *nature* player in game theory terminology.

	$I_2 I_3$	$I_2 O_3$	$O_2 I_3$	$O_2 O_3$
I_1	rq_1	rq_1	rq_1	rq_1
O_1	$f(q_1)$	a_{22}	a_{23}	a_{24}

Table 1 : Allocation cost to Department 1 in Model 1

D1 makes a decision based on the information in the above payoff matrix. It is necessary for D1 to have a criterion for its decision making. It is supposed that each department manager makes their decisions based on the minimax criterion because this criterion is prudent and rational in the case where players face uncertainties.

In general, specifying all the elements in the payoff matrix is necessary to make a decision. Considering this fact, it seems to be difficult that D1 makes a decision based on this payoff matrix. However, as we can conjecture uncertain elements (a_{22}, a_{23}, and a_{24}) roughly by the assumption of the model, we can specify the maximum value in the O_1row of Table 1. It means that D1 can make a decision based on this incomplete payoff matrix.

First, we must determine the maximum value of the each row of Table 1. It is clear that the maximum value of I_1 row is rq_1 The elements in O_1 row have the following relationships.

$$f(q_1) < a_{22} < a_{24} \tag{5}$$
$$f(q_1) < a_{23} < a_{24} \tag{6}$$

D1 knows how to calculate a_{22}, *i.e.* , $a_{22} = \{ f(q_1 + q_3) / (q_1 + q_3)\} q_1$. Furthermore, D1 knows that $f(q)$ is a monotonic and convex function. Therefore, considering $q_1 < q_1 + q_3$, D1 can get the result $f(q_1) < a_{22}$ without specifying the amount q_3. D1 also knows how to calculate a_{24}, *i.e.*, $a_{24} = \{f(\bar{q}) / \bar{q}\} q_1$. From the assumption regarding $f(q)$ and $q_1 + q_3 < q$, D1 can get the result $a_{22} < a_{24}$ without specifying the amount q_2 and q_3. We can gain the inequity (6) with the same procedure as (5). From the inequities (5) and (6), it is clear that the maximum value of O_1 row is a_{24}.

Next we compare rq_1 and a_{24}. As D1 knows how to calculate a_{24}, D1 can conclude that $a_{24} > rq_1 = \{ c(\bar{q}) / \bar{q}\} q_1$ by the assumption $c(\bar{q}) < f(q)$. It means that D1 selects the strategy I_1 in Model 1. If we make the same analysis to D2 and D3, we can obtain the result that D2 selects the strategy I_2 and D3 selects the strategy I_3.

Model 1 illustrates the situation where each department does not have enough information about others, *i.e.*, Model 1 deals with the incomplete information case. Hence each department does not know the response of others to its strategy and make their decision independently. It should be noted that each department can select the strategy I_i without specifying the exact amount of the other department's service demand in Model 1. Though every department does not have complete information about other departments, it is enough for each department to know the allocating formula, the information regarding the cost functions, and $q_j > 0$ (j = 1, 2, 3).

The analysis of Model 1 says that all departments result in obtaining the internal service and a firm can achieve the minimum costs even if the conventional allocation method gives departments disadvantageous allocations. We can justify Proposition in Section 3.2 under the conditions characterized by Model 1.

5.2 Model 2: Coalition Formation Is Possible (1)

We examine the case where departments can form some coalitions and obtain the service jointly. It is supposed that each department can form the coalition which is most favorable to it unrestrictedly. It is also supposed that departments in the same coalition use the same strategy. It means that departments in the same coalition are regarded as one player.

D1 has two alternatives for its coalition formation, *i.e.* a coalition with D2 or a coalition with D3. We examine the case where department 1 forms a coalition with D2, *i.e.*, a coalition structure {{1, 2},{3}}. Coalition {1, 2} is regarded as one player.

Model 2 is the extended case of Model 1 and coalition {1, 2} regard D3 as a *nature*-player, and vice versa. It is supposed that a department tells its service demand to other members in the same coalition. So D1 knows q_2 and D2 knows q_1. But coalition {1, 2} does not know q_3 and D3 does not know q_1 and q_2. Other conditions of Model 2 are the same as those of Model 1.

From the assumption of the model, coalition {1,2 } has two strategies, *i.e.*, I_1I_2 or O_1O_2. If coalition {1, 2 } selects the strategy I_1I_2, the allocated cost to this coalition is $r\,(q_1+q_2)$. Consider the case where coalition {1, 2 } chooses the strategy O_1O_2. If D3 selects the strategy I_3, the charge to coalition {1, 2 } is $f\,(q_1+q_2)$ because coalition {1, 2 } only gets the external service. If D3 selects the strategy O_3, coalition {1, 2} cannot specify the allocated amount because it does not know q_3. We denote this amount as b_{22}. We can obtain the next payoff matrix.

	I_3	O_3
I_1I_2	$r(q_1+q_1)$	$r(q_1+q_1)$
O_1O_2	$f(q_1+q_1)$	b_{22}

Table 2 : Allocated Cost to Coalition {1,2} in Model 2

We have to specify the maximum value in the O_1O_2 row of the Table 2 to make a decision based on this matrix. Coalition {1, 2} knows the allocation formula regarding b_{22}, *i.e.*, b_{22}= { $f\,(\overline{q})\,/\,\overline{q}$}$(q_1+q_2)$. From the assumption of $f\,(q)$ and $q_3>0$, coalition{1, 2} can conclude that $f\,(q_1+q_2)$ without knowing the exact amount of q_3. It is clear that r

$(q_1+q_2)< b_{22}$ by (2). It means that coalition {1, 2} chooses the strategy I_1I_2.

It is supposed that D3 does not know the service amount of coalition {1, 2} and knows that coalition {1, 2} has two strategies, namely, I_1I_2 and O_1O_2. As the assumptions are the same as the coalition {1, 2}, we can get the following payoff matrix D3 faces.

	I_3	O_3
I_1I_2	rq_3	$f(q_3)$
O_1O_2	rq_3	c_{22}

Table 3 : Allocated Cost to Department 3 in Model 2

As D3 knows the property of the cost functions and $q_1+ q_2> 0$, it can conclude that $f(q_3) < c_{22}$ and $rq_3< c_{22}$ without specifying the amount q_1 and q_2. It means that D3 chooses the strategy I_3.

The analysis to the coalition structure {{1, 2},{3}} also applies to the coalition structures {{1, 3},{2}} and {{2, 3},{1}}. It means that all departments obtain the internal service even if they have advantageous opportunities for the external service. We examine the results of Model 2.

Departments can gather more information regarding others by forming some coalitions. A department is able to reduce its uncertainty as to the strategies of other departments if it form a coalition with some departments.

Under the conditions of Model 2, all departments results in obtaining the internal service though each department tries to form some beneficial coalitions to it in order to get the necessary service at a lower cost. We can conclude that Proposition described in Section 3.2 is also justified in Model 2.

5.3 Model 3: Coalition Formation Is Possible (2)

We examine a coalition structure {{1, 2},{3}} as an example. So Model 3 is similar to Model 2 in this regard. Model 3 is definitely different from Model 1 and Model 2 in that it analyzes the case of complete information. Therefore it is supposed that every department knows the necessary amount of the service each other. It is supposed that the other conditions are the same in Model 2.

From the assumption of the model, coalition {1, 2} can estimate the reply of D3 to its strategies (I_1I_2,O_1O_2) and D3 can also estimate the reply of coalition {1, 2} to its strategies (I_3,O_3). We can represent this case as a two-person non-zero-sum non-cooperative game. Thus Model 3 is formulated as a bimatrix game defined by the following two matrix.[25]

[25] The row of the matrix denotes the strategy of coalition {1, 2} and the column of the matrix denotes the strategy of D3.

$$A = \begin{pmatrix} r(q_1+q_2) & r(q_1+q_2) \\ f(q_1+q_2) & \dfrac{f(\bar{q})}{\bar{q}}(q_1-q_2) \end{pmatrix}$$

$$B = \begin{pmatrix} r\,q_3 & f(q_3) \\ r\,q_3 & \dfrac{f(\bar{q})}{\bar{q}}\,q_3 \end{pmatrix}$$

Matrix A and B correspond to Table2 and Table3 respectively. It should be noted that we use some formulas instead of b_{22} and c_{22} in these matrix. It means that coalitio]n {1, 2} and D3 can estimate b_{22} and c_{22} certainly because they know other player's service demand. This is the important point that discriminates between Model 2 and Model 3. It is needless to say that $\{ f(\bar{q}) / \bar{q} \}\ (q_1 + q_2)$ and $\{ f(\bar{q}) / \bar{q} \}\ q_3$ represent the allocated costs to coalition {1, 2} and D3 when all departments get the external service.

We define the set of the mixed strategy of coalition {1,2} and D3.

$$S_{12} = \{ p = (x, 1-x) \mid 0 \le x \le 1 \}$$
$$S_3 = \{ q = (y, 1-y) \mid 0 \le y \le 1 \}$$

S_{12} is a set of the mixed strategy of coalition {1, 2}and S_3 is a set of the mixed strategy of department 3. x is a probability defined on strategy I_1I_2 and y is a probability defined on I_3. So $1-x$ and $1-y$ are probabilities defined on the strategy O_1O_2 and O_3. $E_{12}(p, q)$ and $E_3(p, q)$ are the expected cost of coalition {1, 2} and department 3 respectively.[26]

$$E_{12}(p, q) = E_{12}(x, y) = p\,A\,q^{\mathrm{T}} \tag{7}$$
$$E_3(p, q) = E_3(x, y) = p\,B\,q^{\mathrm{T}} \tag{8}$$

The sufficient and necessary condition that (p, q) is the best reply strategy of coalition {1, 2 } is :

$$E_{12}(p, q) \le E_{12}(1, y) \tag{9}$$
$$E_3(p, q) \le E_{12}(0, y) \tag{10}$$

The subsets of the best reply strategy of coalition {1, 2} are :[27]

$$D_{12,1} = \{(0, y) \mid \alpha < y \le 1\}$$
$$D_{12,2} = \{(x, y) \mid 0 \le x \le 1,\ y = \alpha\}$$
$$D_{12,3} = \{(1, y) \mid 0 \le y < \alpha\}$$

$$\textit{where} \quad \alpha = \frac{\dfrac{f(\bar{q})}{\bar{q}}(q_1+q_2) - r(q_1+q_2)}{\dfrac{f(\bar{q})}{\bar{q}}(q_1+q_2) - f(q_1+q_2)}$$

[26] A superscript T denotes the transpose of the vector.
[27] See Appendix 1.

Therefore the set of the best reply strategy of coalition { 1,2 } is $D_{12} = D_{12,1} \cap D_{12,2} \cap D_{12,3}$. Similarly the best reply strategy of department 3 is $D_3 = D_{3,1} \cap D_{3,2} \cap D_{3,3}$.[28]

$$
\begin{aligned}
D_{3,1} &= \{(x,0) \mid \beta < x \leq 1\} \\
D_{3,2} &= \{(x,y) \mid 0 \leq y \leq 1 ,\, x = \beta\} \\
D_{3,3} &= \{(x,1) \mid 0 \leq x < \beta\} \\
\textit{where} \quad \beta &= \frac{\frac{f(\bar{q})}{\bar{q}} q_3 - r\, q_3}{\frac{f(\bar{q})}{\bar{q}} q_3 - f\, q_3}
\end{aligned}
$$

From the above, the set of the equilibrium points is $D = D_{12} \cap D_3$[29]

$$D = \{\ (O_1 O_2, I_3),\ (\ (\alpha, 1-\alpha\ ), (\beta, 1-\beta)\ ),\ (I_1 I_2, O_3)\ \} \qquad (11)$$

The first equilibrium says that coalition {1,2} chooses the strategy O_1O_2 and D3 chooses the strategy I_3. The overall costs of the firm is $f\,(q_1+ q_2)+c\ (q_3)$. As $f\,(q_1+q_2)+c\ (q_3) \geq c\ (\overline{q})$ by the assumption (3), the minimum costs as a firm is not achieved in this equilibrium.

According to the above reasoning we can also get the results that the minimum cost as a firm is not achieved in the third equilibrium, *i.e.*, (I_1I_2,O_3).

The second equilibrium says that a coalition {1,2} uses the mixed strategy $(\alpha,1-\alpha)$ and D3 uses the mixed strategy $(\beta,\ 1-\ \beta)$. It is clear that $\alpha<1$ and $\beta<1$ by the assumption regarding $f\ (q)$. There is no possibility that the minimum cost as a firm is achieved in this equilibrium.

The results mentioned above apply to the coalition structures {{1,3},{2}} and {{2,3},{1}}. The analysis of Model 3 denotes that the conventional allocation method does not induce the optimal allocation in terms of the firm. We conclude that Proposition in Section 3.2 is not justified in Model 3, namely, in complete information case.

6 Conclusions

In this article, we examine the application of game theory to common cost allocation under the condition that the conventional allocation method is used as an allocation scheme. This approach is different from past studies' approach because it does not intend to propose a new allocation scheme.

Three models are examined in this article. All three models suppose a situation where every department has the incentive to obtain the external service.

Model 1 is different from Model 2 and Model 3 in that it does not admit a coalition

[28] See Appendix 2.

[29] It is clear that D is Nash equilibrium. Because it is the intersection of the best reply strategies of players.

formation. It is assumed that each player does not know the amount of the service other player demands and only knows the possible combination of other player's strategies in Model 1 and Model 2. Thus we formulate these cases as a game in which a *nature* player is presumed. So players make their decision individually in Model 1 and Model 2.

The analysis of Model 1 says that every department obtains the internal service even if the cost of the external service is lower than that of the internal service. We can obtain the same results as Model 1 in the analysis of Model 2. Although we admit the possibility of coalition formation in Model 2, it is concluded that the conventional allocation method does not have serious difficulties in the situation illustrated by Model and Model 2.

While Model 1 and Model 2 treat the case of incomplete information, Model 3 examines the complete information case. Hence, every player knows the complete information to estimate the payoff matrix. Unfortunately, Proposition in Section 3.2 is not justified in Model 3. We must look for another allocation method other than the conventional allocation method in this case. Some cooperative game solutions such as *Shapley* value or *nucleolus* may be promising solution.

The analysis of this article suggests that the conventional allocation method is useful in the situation where departments cannot communicate with other departments in a firm sufficiently. We can discriminate the case where the conventional allocation method is useful and understand that information plays an important role in the common costs allocation setting by the approach proposed in this article.

The results in this article have only limited implications in that the assumptions of the models may not be general and we examine only three departments cases. We have to interpret the assumptions of the models in practice and extend the analysis to n-departments case as a next step.

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Appendix

Appendix 1: The best reply strategy of coalition { 1,2 } in Model 3

$$E_{12}(p, q) = E_{12}(x, y)$$
$$= \left\{ \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - (q_1 + q_2) \right\} x\, y - \left\{ \frac{f(\bar{q})}{\bar{q}} - r \right\} (q_1 + q_2)\, x$$
$$- \left\{ \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - (q_1 + q_2) \right\} y + \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) \tag{12}$$

Insert $x = 1$ and $x = 0$ into (12),

$$E_{12}(1, y) = r\,(q_1 + q_2) \tag{13}$$
$$E_{12}(0, y) = \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - \left\{ \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - (q_1 + q_2) \right\} y \tag{14}$$

Constants a and b are defind as follows. It is clear $a > b$ from the assumpion of the model.

$$a = \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - f\,(q_1 + q_2) \tag{15}$$
$$b = \frac{f(\bar{q})}{\bar{q}} (q_1 + q_2) - r\,(q_1 + q_2) \tag{16}$$

As the sufficient and nessessary condition for $D \in (p,q)$ are (9) and (10), they are denoted by as follows.

$$(1 - x)\,(ay - b) \geq 0 \tag{17}$$
$$x\,(ay - b) \leq 0 \tag{18}$$

The following results are derived from (17) and (18).

$$ay - b > 0 \quad \Rightarrow \quad x = 0 \tag{19}$$
$$ay - b = 0 \quad \Rightarrow \quad 0 \leq x \leq 1 \tag{20}$$
$$ay - b < 0 \quad \Rightarrow \quad x = 1 \tag{21}$$

Equations (19), (20), and (21) correspond to $D_{12,1}$, $D_{12,2}$, and $D_{12,3}$ respectively. Therefore the best reply strategy of coalition {1,2} is $D_{12} = D_{12,1} \cap D_{12,2} \cap D_{12,3}$.

Appendix 2: The best reply strategy of department 3 in Model 3

The derivation of the best reply strategy of department 3 is similar to that of coalition {1,2}. If E3(p,q) is used instead of $E_{12}\,(p, q)$, the following results are derived.

$$cy - d > 0 \quad \Rightarrow \quad y = 0 \tag{22}$$
$$cy - d = 0 \quad \Rightarrow \quad 0 \leq y \leq 1 \tag{23}$$
$$cy - d < 0 \quad \Rightarrow \quad y = 1 \tag{24}$$
$$\textit{where}\ \ c = \frac{f(\bar{q})}{\bar{q}} q_3 - f\,(q_3)$$
$$d = \frac{f(\bar{q})}{\bar{q}} q_3 - r q_3$$

Equation (22), (23), and (24) correspond to $D_{3,1}$, $D_{3,2}$ and $D_{3,3}$ respectively. Therefore the best reply strategy of department 3 is $D_3 = D_{3,1} \cap D_{3,2} \cap D_{3,3}$.